Journal of Applied Mathematics and Mechanics 72 (2008) 192–196

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Localization of eigenforms and limit transformations in problems of the stability of rectangular plates☆

N.V. Banichuk, A.A. Barsuk

Moscow, Russia

ARTICLE INFO

Article history:
Received 29 May 2007

ABSTRACT

Problems of the stability (buckling) of compressed rectangular plates are considered. It is assumed that evenly distributed compressive forces are applied to the opposite hinged edges, while the remaining two edges are not fastened and are free from external actions. It is shown that an increase in the length ratio of the hinged and unfastened sides in close proximity to the free boundary results in the localization of eigenforms corresponding to the lowest eigenvalues.

In problems of the elastic stability (buckling) of infinitely long rectangular plates (panels) hinged along their long sides and exposed to unvarying compressive forces over their length, it is normal to assume that the deflection function corresponds to a cylindrical surface. This enables the solution of the two-dimensional problem of the loss of stability of a narrow strip to be replaced by the solution of the corresponding one-dimensional problem. However, this replacement, as first shown by Ishlinskii[1] (see also Refs. [2–4]), is not entirely rigorous and does not result in an adequate description of the loss of stability of long rectangular plates. An analysis of the analytical solution of the two-dimensional problem for a plate with a finite side length ratio and a subsequent limit transition as the plate length tends to infinity indicate that the critical force of loss of stability (the minimum eigennumber) differs by a finite amount from the critical force of compression that is obtained in the one-dimensional problem. A similar feature of the limit transition and a discrepancy between the solutions of the two- and one-dimensional problems can be seen in the problem of the free oscillations of a plate under the same conditions of fastening of its boundaries.[4]

The boundary-value problems of the loss of stability of long rectangular plates investigated, represented in dimensionless form, contain a small parameter at the highest derivative and can be classified as singularly perturbed boundary-value problems[5]; similar problems were considered earlier.[6] The solutions of these problems are characterized by an abrupt change in close proximity to the boundaries. In singularly perturbed eigenvalue problems of loss of elastic stability, the behaviour of the eigenforms can be expected to possess similar features.

Below, the problems, considered earlier,[4] of loss of stability and of free oscillations of elastic rectangular plates are investigated, and an asymptotic analysis of the solutions obtained is carried out. This enables additional features of the spectrum of eigenvalues and the behaviour of eigenforms as a function of the plate side ratio to be established.

1. Formulation of the problem of elastic stability and presentation of the solutions

The stability of an elastic rectangular plate with free lateral sides of length l and hinged sides of length $2b$ will be examined. Evenly distributed compressive forces of magnitude p are applied to the hinged edges. Finding the critical load at which the plate loses its stability and bulges reduces to finding the minimum eigenvalue p and the corresponding eigenfunction (form of loss of stability) $w = w(x, y)$ from the solution of the following eigenvalue boundary-value problem

$$D\Delta^2 w + p w_{xx} = 0, \quad 0 \le x \le l, \quad -b \le y \le b \tag{1.1}$$

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 2, pp. 302–307, 2008.

E-mail address: banichuk@ipmnet.ru (N.V. Banichuk).

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2008.04.005

$$w(0,y) = w_{xx}(0,y) = w(l,y) = w_{xx}(l,y) = 0, \quad -b \le y \le b \tag{1.2}$$

$$w_{yy}(x,\pm b) + \upsilon w_{xx}(x,\pm b) = 0, \quad w_{yyy}(x,\pm b) + (2-\nu)w_{xxy}(x,\pm b) = 0, \quad 0 \le x \le l \tag{1.3}$$

for the equation of the lateral flexure (1.1) with the boundary conditions of support (1.2) and the boundary conditions of the absence of moments and shearing-forces on the free edges (1.3). Here, $D = Eh^3/[12(1-\upsilon^2)]$ is the cylindrical stiffness of the plate, h is its thickness, E and υ are Young's modulus and Poisson's ratio of the material and Δ^2 is a biharmonic operator. The subscripts x and y denote partial derivatives with respect to the corresponding variables.

The solution of problem (1.1)–(1.3) will be sought in the form[1]

$$w(x,y) = f(y)\sin\frac{\pi x}{l} \tag{1.4}$$

where $f(y)$ is the desired function. Eq. (1.4) shows that the function $w(x, y)$ satisfies the boundary conditions of support (1.2). The boundary-value problem for finding the function $f(y)$ is obtained by substituting expression (1.4) into Eq. (1.1) and conditions (1.3), and in dimensionless quantities

$$y = b\tilde{y}, \quad \mu = \gamma^2 = pl^2/(\pi^2 D), \quad \varepsilon = l/(\pi b) \tag{1.5}$$

(the tilde is omitted below) is written in the form

$$\varepsilon^4 f'''' - 2\varepsilon^2 f'' + (1-\mu)f = 0, \quad -1 \le y \le 1$$

$$\varepsilon^2 f''(\pm 1) - \upsilon f(\pm 1) = 0, \quad \varepsilon^3 f'''(\pm 1) - (2-\upsilon)\varepsilon f'(\pm 1) = 0 \tag{1.6}$$

The eigenvalue boundary-value problem (1.6) is invariant under to the operation of symmetry $y \to -y$, and consequently the eigenforms (EFs) can be classified according to evenness:

$$f^s(y) = f^s(-y), \quad f^a(y) = -f^a(-y), \quad 0 \le y \le 1 \tag{1.7}$$

where f^s and f^a are symmetric and antisymmetric EFs about to the x axis.

Using relation (1.6), we obtain general representations for the functions $f^s(y)$ and linear algebraic equations for determining the constants A^s and B^s

$$f^s(y) = A^s \operatorname{ch}\frac{\kappa_+ y}{\varepsilon} + B^s \operatorname{ch}\frac{\kappa_- y}{\varepsilon}; \quad \kappa_\pm = \sqrt{1 \pm \gamma} \tag{1.8}$$

$$A^s(\kappa_+^s - \upsilon)\operatorname{ch}\frac{\kappa_+}{\varepsilon} + B^s(\kappa_-^s - \upsilon)\operatorname{ch}\frac{\kappa_-}{\varepsilon} = 0, \quad -A^s\kappa_+(\kappa_-^s - \upsilon)\operatorname{sh}\frac{\kappa_+}{\varepsilon} - B^s\kappa_-(\kappa_+^s - \upsilon)\operatorname{sh}\frac{\kappa_-}{\varepsilon} = 0 \tag{1.9}$$

The conditions for a non-trivial solution to exist in the form of (1.8) and (1.9), which reduce to the determinant of the homogeneous system (1.9) vanishing, are written in the form of a transcendental equation

$$\Delta^s(\gamma,\varepsilon) = \kappa_-(\kappa_+^s - \upsilon)^2 \operatorname{ch}\frac{\kappa_+}{\varepsilon}\operatorname{sh}\frac{\kappa_-}{\varepsilon} - \kappa_+(\kappa_-^s - \upsilon)^2 \operatorname{sh}\frac{\kappa_+}{\varepsilon}\operatorname{ch}\frac{\kappa_-}{\varepsilon} = 0$$

$$\kappa_+ = \kappa_+(\gamma), \quad \kappa_- = \kappa_-(\gamma) \tag{1.10}$$

which can be used to determine the eigenvalues $\mu = \gamma^2$ corresponding to symmetric EFs with different values of the parameter $\varepsilon = \lambda/(\pi b)$.

Similarly, using relations (1.6) and (1.7), we can obtain representations for antisymmetric EFs $f^a(y)$, equations for determining the corresponding constants A^a and B^a and the transcendental equation $\Delta^a(\gamma, \varepsilon) = 0$ for determining the eigenvalues corresponding to antisymmetric EFs. These representations and equations differ from (1.8)–(1.10) in the replacement ch → sh and sh → ch.

2. Asymptotic analysis of symmetric eigenforms and corresponding eigenvalues

Below we will investigate the solutions of eigenvalue problem (1.6) for long plates, i.e. for small values of the parameter ε, and the limit transition as $\varepsilon \to 0$. Note that when $0 \le \varepsilon \ll 1$ and for any $\gamma \ge 0$

$$\kappa_+/\varepsilon = \sqrt{1+\gamma}/\varepsilon \gg 1 \tag{2.1}$$

and the transcendental equation for determining the eigenvalues (1.10), taking into account the asymptotic representation

$$\operatorname{ch}\frac{\kappa_+}{\varepsilon} = \operatorname{sh}\frac{\kappa_+}{\varepsilon} \approx \frac{1}{2}\exp\frac{\kappa_+}{\varepsilon}$$

(which holds for large values of the argument), will take the form

$$\Delta^s(\gamma,\varepsilon) = \kappa_-(\kappa_+^s - \upsilon)^2 \operatorname{th}\frac{\kappa_-}{\varepsilon} - \kappa_+(\kappa_-^s - \upsilon)^2 = 0 \tag{2.2}$$

We will investigate separately the versions of arrangement of the eigenvalues in the intervals $0 \le \gamma \le 1$ and $\gamma > 1$.

We will first consider the case where $0 \le \gamma \le 1$. Note that, when $\upsilon \ne 0$, the value $\gamma = 1$ does not satisfy Eq. (1.9), i.e., it is not the eigenvalue corresponding to the symmetric eigenfunction. The equation $\Delta^s(\gamma, \varepsilon) = 0$ defines a certain dependence $\gamma(\varepsilon)$ and thereby also determines the limit behaviour of the function $\kappa_-(\varepsilon)/\varepsilon = \sqrt{1-\gamma(\varepsilon)}/\varepsilon$ as $\varepsilon \to 0$.

We will consider three versions of the limit behaviour of this function:

$$1)\ \lim_{\varepsilon \to 0} \frac{\kappa_-(\varepsilon)}{\varepsilon} = 0, \quad 2)\ \lim_{\varepsilon \to 0} \frac{\kappa_-(\varepsilon)}{\varepsilon} = C \ne 0, \quad 3)\ \lim_{\varepsilon \to 0} \frac{\kappa_-(\varepsilon)}{\varepsilon} = \infty$$

For the first two versions we have

$$\lim_{\varepsilon \to 0} \gamma(\varepsilon) = 1$$

which contradicts Eq. (2.2). The third version means that

$$\lim_{\varepsilon \to 0} \gamma(\varepsilon) = \gamma_0, \quad \gamma_0 \ne 1$$

and here $\gamma(\varepsilon) \le \gamma_0$ when $\varepsilon \ge 0$ (see Ref. 4). To find the value of γ with exponential accuracy, we have $\text{th}(\kappa_-/\varepsilon) \approx 1$, and as a result we arrive at an algebraic equation, the solution of which[4] is

$$\mu = \gamma_0^2 = (1-\upsilon)[3\upsilon - 1 + 2(1 - 2\upsilon(1-\upsilon))^{1/2}] \tag{2.3}$$

The value of γ_0 found earlier[1] turns out to be close to unity and lower than for antisymmetric forms.[4] For the values of Poisson's ratio $\upsilon = 0.2$, 0.3 and 0.5, we have $\mu = \gamma_0^2 = 0.9994$, 0.9962 and 0.9571.

The symmetric EFs $f^s(y)$ corresponding to the eigenvalues $\gamma(\varepsilon) \le \gamma_0 < 1$ are obtained using relations (1.8) and (1.9). We have

$$f^s(y) = C\left[\frac{\upsilon - \kappa_-^2}{\kappa_+^2 - \upsilon}\frac{\text{ch}\dfrac{\kappa_+ y}{\varepsilon}}{\text{ch}\dfrac{\kappa_+}{\varepsilon}} + \frac{\text{ch}\dfrac{\kappa_- y}{\varepsilon}}{\text{ch}\dfrac{\kappa_-}{\varepsilon}}\right] \tag{2.4}$$

where C is an arbitrary constant. Since

$$\kappa_+/\varepsilon \gg 1, \quad \kappa_-/\varepsilon \gg 1 \ \text{при}\ 0 \le \varepsilon \ll 1$$

expression (2.4) can be represented with exponential accuracy in the form

$$f^s(y) = C\left[\frac{\upsilon - \kappa_-^2}{\kappa_+^2 - \upsilon}\exp\left(-\frac{\kappa_+}{\varepsilon}(1-y)\right) + \exp\left(-\frac{\kappa_-}{\varepsilon}(1-y)\right)\right], \quad 0 \le y \le 1$$

$$f^s(y) = f^s(-y) \ \text{при}\ -1 \le y \le 0 \tag{2.5}$$

Thus, as ε decreases, localization of symmetric EFs occurs in the vicinity of the free sides, i.e., as $y \to \pm 1$.

Now consider the case where $\gamma > 1$. The condition for determining the eigenvalues and the expression for the corresponding symmetric EFs are obtained from relations (1.8), (1.10) and (2.4), taking into account the relations

$$r_\pm = \sqrt{\gamma \pm 1}, \quad \kappa_+ = r_+, \quad \kappa_- = ir_-; \quad \text{sh}\frac{\kappa_- y}{\varepsilon} = i\sin\frac{r_- y}{\varepsilon}, \quad \text{ch}\frac{\kappa_- y}{\varepsilon} = \cos\frac{r_- y}{\varepsilon}$$

and are written in real form

$$\Delta^s(\gamma, \varepsilon) = r_-(r_+^2 - \upsilon)^2\,\text{ch}\frac{r_+}{\varepsilon}\sin\frac{r_-}{\varepsilon} + r_+(r_-^2 + \upsilon)^2\,\text{sh}\frac{r_+}{\varepsilon}\cos\frac{r_-}{\varepsilon} = 0 \tag{2.6}$$

$$f^s(y, \varepsilon) = C\left[\frac{r_-^2 + \upsilon}{r_+^2 - \upsilon}\frac{\text{ch}\dfrac{r_+ y}{\varepsilon}}{\text{ch}\dfrac{r_+}{\varepsilon}}\cos\frac{r_-}{\varepsilon} + \cos\frac{r_- y}{\varepsilon}\right] \tag{2.7}$$

For small values of ε, Eq. (2.6) can be represented with exponential accuracy in the form

$$\Delta^s(\gamma, \varepsilon) = r_-(r_+^2 - \upsilon)^2\sin\frac{r_-}{\varepsilon} + r_+(r_-^2 + \upsilon)^2\cos\frac{r_-}{\varepsilon} = 0 \tag{2.8}$$

We will obtain an asymptotic representation for the relation $\gamma(\varepsilon)$ defined by Eq. (2.8). To this end, we introduce a new variable τ:

$$r_- = \sqrt{\gamma - 1} = \tau\varepsilon, \quad \gamma = 1 + \tau^2\varepsilon^2 \tag{2.9}$$

and transform Eq. (2.8) in the following way

$$\varepsilon\tau(2 - \upsilon + \varepsilon^2\tau^2)^2\sin\tau + \sqrt{2 + \varepsilon^2\tau^2}(\upsilon + \varepsilon^2\tau^2)^2\cos\tau = 0 \tag{2.10}$$

Note that $\tau = 0$ does not satisfy Eq. (2.10), and when $\varepsilon = 0$ this equation has the solutions $\tau_k = (2k-1)\pi/2$ ($k = 1, 2, \ldots$), in accordance with which the solution of Eq. (2.10) is found in the form

$$\tau_k(\varepsilon) = (2k-1)\frac{\pi}{2} + \delta_k(\varepsilon), \quad \delta_k(0) = 0, \quad 0 \le \delta_k(\varepsilon) < \frac{\pi}{4} \tag{2.11}$$

We will determine the asymptotic behaviour of $\delta_k(\varepsilon)$ for small values of the parameter ε ($0 \le \varepsilon \ll 1$) by substituting expression (2.1) into Eq. (2.10). We will then have

$$\delta_k(\varepsilon) \approx (2k-1)\frac{\pi}{2}\alpha_1\varepsilon + \ldots, \quad \alpha_1 = \frac{(2-\upsilon)^2}{\sqrt{2}\upsilon^2}, \quad 0 \le \varepsilon \ll 1$$

Thus, we arrive at the following expression for the dependence of the eigenvalue γ_k on the parameter ε when $\varepsilon \ll 1$

$$\gamma_k(\varepsilon) = 1 + \varepsilon^2\tau_k^2(\varepsilon) \approx 1 + (2k-1)^2\frac{\pi^2}{4}\varepsilon^2 + (2k-1)^2\frac{\pi^2}{2}\alpha_1\varepsilon^3 + \ldots \tag{2.12}$$

The expression for the symmetric EFs corresponding to eigenvalues $\gamma_k(\varepsilon)$ can be obtained with any required accuracy by substituting expressions (2.9) and (2.11) into Eq. (2.7). Making asymptotic estimates with an accuracy to terms $O(\varepsilon^2)$, we arrive at the following asymptotic representations for the EFs

$$f_k^s(y, \varepsilon) = C\left[\frac{r_-^2 + \upsilon}{r_+^2 - \upsilon}\exp\left(-\frac{\sqrt{2}(1-y)}{\varepsilon}\right)\alpha_1(2k-1)\frac{\pi}{2}(-1)^{k-1}\varepsilon + \cos\left((2k-1)\frac{\pi}{2}y\right)\right.$$
$$\left.-\varepsilon\alpha_1(2k-1)\frac{\pi}{2}y\sin\left((2k-1)\frac{\pi}{2}y\right)\right], \quad 0 \le y \le 1$$
$$f^s(y) = f_k^s(-y), \quad -1 \le y \le 0 \tag{2.13}$$

The first term in the square brackets decays exponentially with distance from the free edges of the plate ($y = \pm 1$). Note also that, for $k = 1$, in the zero approximation with respect to ε, the EF has the form

$$f_1^s(y, 0) = C\cos\frac{\pi y}{2} \tag{2.14}$$

which corresponds to the EF of the loss of stability of a plate hinged along its lateral sides.

3. Asymptotic analysis of antisymmetric eigenforms and corresponding eigenvalues

We will now consider the equation for determining the eigenvalues corresponding to antisymmetric EFs. For small values of ε, i.e., when condition (2.1) is satisfied, this equation is written approximately in a form differing from (2.2) in that $\text{th}(\kappa_-/\varepsilon)$ is replaced by $\text{cth}(\kappa_-\varepsilon)$.

The subsequent analysis is similar to the analysis of symmetric EFs and the corresponding eigenvalues (see Section 2). We will give the final results.

An antisymmetric EF is localized in close proximity to the edges $y = \pm 1$ in a similar manner to symmetric EFs when $0 \le \gamma(\varepsilon) \le 1$. When $\gamma > 1$ we obtain an asymptotic representation of the relation $\gamma(\varepsilon)$ defined by the equation $\Delta^a(\gamma, \varepsilon) = 0$. For this, we introduce a new variable τ according to Eq. (2.9) and write the equation in the form

$$\varepsilon\tau(2 - \upsilon + \varepsilon^2\tau^2)^2\cos\tau + \sqrt{2 - \varepsilon^2\tau^2}(\upsilon + \varepsilon^2\tau^2)^2\sin\tau = 0 \tag{3.1}$$

Note that a zero EF corresponds to the solution $\tau = 0$. A non-zero solution of Eq. (3.1) is found in the form

$$\tau_k(\varepsilon) = k\pi + \delta(\varepsilon), \quad k = 1, 2, \ldots$$

When $\varepsilon = 0$, Eq. (3.1) has the solution $\tau_k(0) = k\pi$ ($k = 1, 2, \ldots$), and consequently $\delta_k(0) = 0$ ($k = 1, 2, \ldots$). For small values of ε, from Eq. (3.1), with an accuracy up to linear terms in ε, we will have

$$\delta_k(\varepsilon) \approx k\pi\alpha_1\varepsilon + \ldots, \quad \alpha_1 = \frac{(2-\upsilon)^2}{\sqrt{2}\upsilon^2}, \quad 0 \le \varepsilon \ll 1 \tag{3.2}$$

When $k = 1$, we have $\tau_1(\varepsilon) \approx \pi$ and $\gamma^a(\varepsilon) = 1 + (\tau^a(\varepsilon))^2 \approx 1 + \pi^2\varepsilon^2$, while for a symmetric EF

$$\gamma_1^s(\varepsilon) \approx 1 + \frac{\pi^2}{4}\varepsilon^2 < \gamma_1^a(\varepsilon) \tag{3.3}$$

In conclusion, note that all the relations obtained in analysing the stability remain unchanged in the problem of free transverse oscillations of a rectangular elastic plate with the appropriate replacement of the parameters of the problem. Consequently, in the case of elastic vibrations also, localization of the eigenforms in the vicinity of the free boundaries occurs.

Acknowledgements

This research was financed by the Russian Foundation for Basic Research (05-08-18094, 08-08-00025-a) and by the Programme of Support for the Leading Scientific Schools (NSh-169.2008.1).

References

1. Ishlinskii A Yu. On a limit transformation in the theory of stability of rectangular plates. *Dokl Akad Nauk SSSR* 1954;**95**(3):477–9.
2. Ishlinskii A Yu. *Applied Problems of Mechanics*, Vol. 2. Moscow: Nauka; 1986.
3. Donnell LH. *Beams, Plates and Shells*. New York: McGraw-Hill; 1976.
4. Banichuk NV, Ishlinskii A Yu. Certain features of the problems of the stability and oscillations of rectangular plates. *Prikl Mat Mekh* 1995;**59**(4):620–5.
5. Nayfen AN. *Perturbation Methods*. New York: Wiley; 1973.
6. Tovstik PE. *Stability of Thin Shells*. Moscow: Nauka/Fizmatlit; 1995.

Translated by P.S.C.